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END STRESS CALCULATIONS ON ELASTIC CYLINDERS

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Abstract—For a semi-infinite circular elastic cylinder $z \geq 0$, $r \leq 1$ deformed solely by a distribution of stress and displacements on its flat end $z = 0$, the Love stress function can be expanded in a series of eigenfunctions of known form. For problems in which suitable mixed stress and displacements boundary conditions are prescribed on $z = 0$ the coefficients appearing in the expansion can be determined in an explicit form via sets of biorthogonal functions. When normal and shear stresses are prescribed on $z = 0$ no such closed expressions for the coefficients exist and approximate methods usually lead to infinite systems of linear equations which are solved by truncation. Stability of solution as the order of truncation is increased can only be guaranteed theoretically when the infinite matrix is diagonally dominated, and this is not the case for existing methods. A Galerkin method has been developed using weighting functions chosen so as to optimise the diagonal dominance of the infinite matrix, and numerical results show that, although the resulting matrix is not completely diagonally dominated, the resulting coefficients show an improvement in stability in the sense that they do not change significantly as the order of truncation is increased.

1. INTRODUCTION

The Love stress function $\Phi(r, z)$ in an elastic cylinder $z \geq 0$, $r \leq 1$ subjected to homogeneous boundary conditions on the curved boundary $r = 1$ can be expressed as an eigenfunction expansion of the form

$$\sum_n c_n \phi(r; \lambda_n) \exp(-\lambda_n z) \tag{1.1}$$

where λ_n is an eigenvalue determined from the conditions on $r = 1$. For the case of a traction-free curved face, λ_n is a root of

$$\lambda^2\{J_0^2(\lambda) + J_1^2(\lambda)\} = 2(1-\nu)J_1^2(\lambda). \tag{1.2}$$

Little and Childs[1] have given a construction for determining the coefficients c_n in the expansion (1.1) for cases in which the data on the flat end $z = 0$ takes the form of prescribed values of either of the pairs

$$\sigma_{zz} \text{ and } u_r$$

or (1.3)

$$\sigma_{rz} \text{ and } u_z.$$

For these "canonical" problems the $\{c_n\}$ are found explicitly as quadratures of the data with appropriate biorthogonal functions derived from the $\phi(r; \lambda_n)$.

In the present paper we consider the problem of determining the coefficients when σ_{zz} and σ_{rz} are prescribed. It is known that no explicit solution exists for this case, and the $\{c_n\}$ must be found by approximate methods leading in general to infinite matrices which can only be inverted in truncated form.

This problem has already been studied extensively for the elastic strip, $x \geq 0$, $|y| \leq 1$. Spence[2] introduced a set of weighting functions derived from members of the family of

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biorthogonal functions, which in the case of the traction problem for the strip, namely

$$\sigma_{xx}, \sigma_{xy} \text{ prescribed on } x = 0$$

lead to a diagonally dominated system of equations

$$\sum_n A_{mn} c_n = d_m \tag{1.4}$$

where $\mathbf{A} = \mathbf{I} - \mathbf{G}$, with the row sum norm $\|\mathbf{G}\| < 1$. For such a system, the solution $\mathbf{c}^{(N)}$ say, of the truncated system

$$\sum_{n=1}^{N} A_{mn}^{(N)} c_n^{(N)} = d_m^{(N)} \tag{1.5}$$

is known to converge to the solution of the full system as $N \to \infty$, and this was borne out for the cases tested, in which it was found that changing the order of truncation N did not lead to significant changes in the coefficients. This was not found to be the case with other published methods that were tested.

2. THE NEW FORMULATION

The construction given by Little and Childs[1] for obtaining biorthogonal functions for the two canonical end problems for the elastic cylinder, thus enabling them to obtain the coefficients appearing in (1.1) explicitly, has not proved to be the most suitable for the present studies. The main disadvantage is that for the stress problem it is not possible to "optimise" the weighting functions, thus improving the diagonal dominance of the infinite matrix arising in this problem. Consequently we choose a different but equivalent set of four stress- and displacement-related variables which will be prescribed on $z = 0$.

In terms of the biharmonic "Love" stress function (Love[3], Art. 188) the stresses and displacements are given by

$$\sigma_{rr} = \frac{\partial}{\partial z}\left\{\nu\nabla^2\Phi - \frac{\partial^2\Phi}{\partial r^2}\right\} \qquad \sigma_{rz} = \frac{\partial}{\partial r}\left\{(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right\} \tag{2.1,2}$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left\{(2-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right\} \qquad \sigma_{\theta\theta} = \frac{\partial}{\partial z}\left\{\nu\nabla^2\Phi - \frac{1}{r}\frac{\partial\Phi}{\partial r}\right\} \tag{2.3,4}$$

$$2\mu u_r = -\frac{\partial^2\Phi}{\partial r\partial z} \qquad 2\mu u_z = 2(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2} \tag{2.5,6}$$

where ν is Poisson's ratio and

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \equiv B^2 + \frac{\partial^2}{\partial z^2}. \tag{2.7}$$

If the cylinder is subjected to stress-free side conditions on $r = 1$ and a self-equilibrating distribution of stresses and displacements on $z = 0$ then Φ may be expanded as an eigenfunction expansion

$$\Phi(r, z) = \sum_n c_n \phi(r; \lambda_n)\, e^{-\lambda_n z} \tag{2.8}$$

where λ_n is a root of

$$\lambda^2\{J_0^2(\lambda) + J_1^2(\lambda)\} - 2(1-\nu)J_1^2(\lambda) = 0, \tag{2.9}$$

and

$$\phi(r; \lambda) = [2(1-\nu)J_1(\lambda) + \lambda J_0(\lambda)]J_0(\lambda r) + \lambda J_1(\lambda) r J_1(\lambda r). \tag{2.10}$$

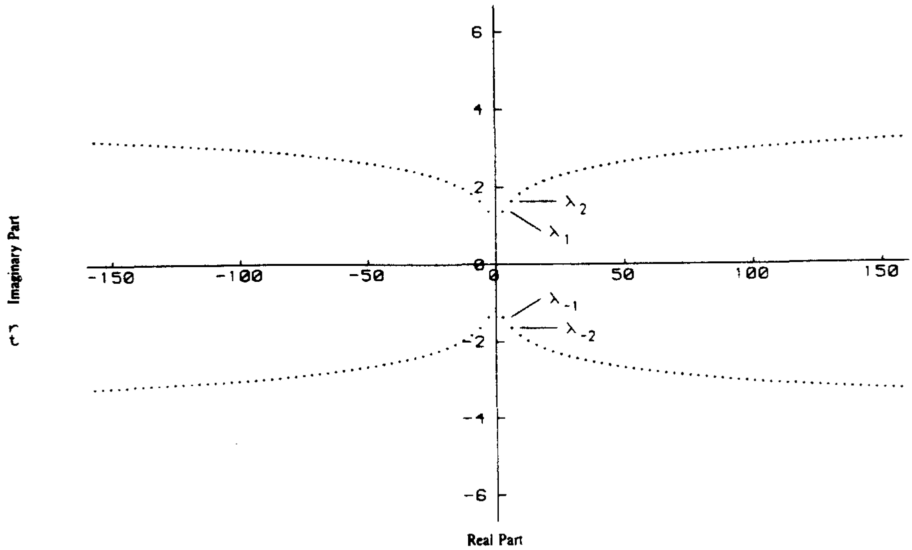

Fig. 1. First 50 cylinder eigenvalues.

The correct interpretation of the summation (2.8) is obtained by numbering the roots of (2.9) in the right half-plane so that $\lambda_{-n} = \overline{\lambda_n}$ (see Fig. 1) and writing the expansion more precisely as

$$\Phi(r, z) = \sum_{n=-\infty}^{n=+\infty}{}' \, c_n \phi(r; \lambda_n) \exp(-\lambda_n z) \tag{2.11}$$

where the prime indicates that the term with $n = 0$ does not appear in the summation. This implies that the normal stress distribution is equilibrated. i.e. $\int_0^1 r\sigma_{zz}(r, 0)\,\mathrm{d}r = 0$.

The present choise of prescribed functions together with their expansions in terms of the "derived" functions $\phi_n^{(\alpha)}(r)$ are given by

$$\begin{bmatrix} f^{(1)}(r) \\ f^{(2)}(r) \\ f^{(3)}(r) \\ f^{(4)}(r) \end{bmatrix} = \begin{bmatrix} \partial\sigma_{zz}/\partial r \\ \sigma_{rz} \\ -(1-2\nu)\dfrac{\partial}{\partial r} B^2\phi_z + 2\nu\Phi_{zzzr} \\ (1+\nu)\dfrac{\partial}{\partial r}\nabla^2\Phi \end{bmatrix}_{z=0} = \sum c_n \begin{bmatrix} \phi_n^{(1)}(r) \\ \phi_n^{(2)}(r) \\ \phi_n^{(3)}(r) \\ \phi_n^{(4)}(r) \end{bmatrix} \tag{2.12}$$

This can be seen to be equivalent to prescribing the unmodified stresses and displacements as in Little and Childs—e.g. if σ_{zz} and u_r are known on $z = 0$, then so are $f^{(1)}$ and $f^{(3)}$ as defined above.

In terms of $\phi(r; \lambda)$ the derived functions $\phi_n^{(\alpha)}$ are given by

$$\phi_n^{(1)}(r) = -\lambda_n\left\{(2-\nu)\frac{\mathrm{d}}{\mathrm{d}r}B^2\phi + (1-\nu)\lambda_n^2 \cdot \frac{\mathrm{d}\phi}{\mathrm{d}r}\right\} \tag{2.13}$$

$$\phi_n^{(2)}(r) = (1-\nu)\frac{\mathrm{d}}{\mathrm{d}r}B^2\phi - \nu\lambda_n^2 \cdot \frac{\mathrm{d}\phi}{\mathrm{d}r} \tag{2.14}$$

$$\phi_n^{(3)}(r) = -\lambda_n\left\{-(1-2\nu)\frac{\mathrm{d}}{\mathrm{d}r}B^2\phi + 2\nu\lambda_n^2 \cdot \frac{\mathrm{d}\phi}{\mathrm{d}r}\right\} \tag{2.15}$$

$$\phi_n^{(4)}(r) = (1+\nu)\left\{\frac{\mathrm{d}}{\mathrm{d}r}B^2\phi + \lambda_n^2 \cdot \frac{\mathrm{d}\phi}{\mathrm{d}r}\right\} \tag{2.16}$$

and explicit expressions for these functions in terms of Bessel functions are

$$\phi_n^{(1)}(r) = \lambda_n^4\{\lambda_n J_1(\lambda_n) r J_0(\lambda_n r) + [2J_1(\lambda_n) - \lambda_n J_0(\lambda_n)]J_1(\lambda_n r)\} \quad (2.17)$$

$$\phi_n^{(2)}(r) = \lambda_n^4\{-J_1(\lambda_n) r J_0(\lambda_n r) + J_0(\lambda_n) J_1(\lambda_n r)\} \quad (2.18)$$

$$\phi_n^{(3)}(r) = \lambda_n^4\{-\lambda_n J_1(\lambda_n) r J_0(\lambda_n r) + [2\nu J_1(\lambda_n) + \lambda_n J_0(\lambda_n)]J_1(\lambda_n r)\} \quad (2.19)$$

$$\phi_n^{(4)}(r) = -2(1+\nu)\lambda_n^3 J_1(\lambda_n) J_1(\lambda_n r). \quad (2.20)$$

3. DERIVATION OF BIORTHOGONAL FUNCTIONS

$\phi(r;\lambda)$ is a solution of the reduced biharmonic equation

$$\left[\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}+\lambda^2\right]^2\phi = 0, \quad (3.1)$$

and as in Spence[2] this equation may be expressed as a matrix differential equation in either $\phi_m^{(1)}$ and $\phi_m^{(3)}$ or $\phi_m^{(2)}$ and $\phi_m^{(4)}$†. For the (1, 3)-canonical problem the matrix equation

$$\begin{bmatrix} -\nu\left[B^2-\frac{1}{r^2}\right] & \left[B^2-\frac{1}{r^2}\right] \\ -\left[B^2-\frac{1}{r^2}\right] & -(2+\nu)\left[B^2-\frac{1}{r^2}\right]\end{bmatrix}\begin{bmatrix}\phi_m^{(1)} \\ \phi_m^{(3)}\end{bmatrix} = (1+\nu)\lambda_m^2\begin{bmatrix}\phi_m^{(1)} \\ \phi_m^{(3)}\end{bmatrix} \quad (3.2)$$

can readily be shown using (2.13, 15) to reduce to

$$\frac{d}{dr}\{B^2+\lambda^2\}^2\phi = 0. \quad (3.3)$$

The condition $\sigma_{rz} = 0$ on $r = 1$ may be written in terms of $\phi_m^{(1)}$ and $\phi_m^{(3)}$ as

$$\nu\phi_m^{(1)}(1) - \phi_m^{(3)}(1) = 0. \quad (3.4)$$

The corresponding boundary condition for σ_{rr} is

$$(1-\nu)D\phi_m^{(1)}(1) + 2D\phi_m^{(3)}(1) + (1+\nu)\phi_m^{(3)}(1) = 0 \quad (3.5)$$

where $D \equiv d/dr$. The derivatives of $\phi_m^{(1)}$ and $\phi_m^{(3)}$ contain the fourth derivatives of ϕ, and in obtaining (3.5) it has been necessary to use the reduced biharmonic equation (3.1) evaluated at $r = 1$ to express the σ_{rr} condition in the required form.

As in Spence[2] the functions $\psi_n^{(1)}$ and $\psi_n^{(3)}$ which are biorthogonal to $\phi_m^{(1)}$ and $\phi_m^{(3)}$ are obtained as the eigenfunctions of the differential operator adjoint to (3.2).‡ If $\psi_n^{(1,3)}$ satisfies the equation

$$\begin{bmatrix} -\nu\left[B^2-\frac{1}{r^2}\right] & -\left[B^2-\frac{1}{r^2}\right] \\ \left[B^2-\frac{1}{r^2}\right] & -(2+\nu)\left[B^2-\frac{1}{r^2}\right]\end{bmatrix}\begin{bmatrix}\psi_n^{(1)} \\ \psi_n^{(3)}\end{bmatrix} = (1+\nu)\lambda_n^2\begin{bmatrix}\psi_n^{(1)} \\ \psi_n^{(3)}\end{bmatrix} \quad (3.6)$$

(which is the transpose of (3.2)) with associated boundary conditions

$$\psi_n^{(1)}(1) + \nu\psi_n^{(3)}(1) = 0 \quad (3.7)$$

$$D\psi_n^{(3)}(1) - \psi_n^{(1)}(1) = 0, \quad (3.8)$$

†This is another advantage of the present formulation—the Little and Childs derived functions do not appear to be the solutions of any underlying matrix differential equation.

‡The construction of biorthogonal functions for the (2, 4)-problem is a modification of the work of Klemm[4], Klemm and Little[5] who treated the full non-axisymmetric end loading problem. Putting $\theta \equiv 0$, $\partial/\partial\theta \equiv 0$ in their construction gives the biorthogonality given here. However, their construction for the (1, 3)-problem does not lead to a pure biorthogonality from which the coefficients can be determined explicitly, and the construction described below is new.

then if $\langle(*)\rangle$ denotes $\int_0^1 (*) \, . \, r\,\mathrm{d}r$

$$(\lambda_m^2 - \lambda_n^2)\langle \phi_m^{(1)}\psi_n^{(1)} + \phi_m^{(3)}\psi_n^{(3)}\rangle = 0 \tag{3.9}$$

and consequently

$$\langle \phi_m^{(1)}\psi_n^{(1)} + \phi_m^{(3)}\psi_n^{(3)}\rangle = 0 \text{ for } m \neq n. \tag{3.10}$$

The same construction may be obtained for the (2, 4)-canonical problem. This time the required matrix differential equation is

$$\begin{bmatrix} -(1+\nu)\left[B^2 - \frac{1}{r^2}\right] & \left[B^2 - \frac{1}{r^2}\right] \\ 0 & -(1+\nu)\left[B^2 - \frac{1}{r^2}\right] \end{bmatrix} \begin{bmatrix} \phi_m^{(2)} \\ \phi_m^{(4)} \end{bmatrix} = (1+\nu)\lambda_m^2 \begin{bmatrix} \phi_m^{(2)} \\ \phi_m^{(4)} \end{bmatrix} \tag{3.11}$$

with corresponding boundary conditions

$$\phi_m^{(2)}(1) = 0 \tag{3.12}$$

$$(1+\nu)D\phi_m^{(2)}(1) = D\phi_m^{(4)}(1) + \nu\phi_m^{(4)}(1) \tag{3.13}$$

and the adjoint equation and boundary conditions are

$$\begin{bmatrix} -(1+\nu)\left[B^2 - \frac{1}{r^2}\right] & 0 \\ \left[B^2 - \frac{1}{r^2}\right] & -(1+\nu)\left[B^2 - \frac{1}{r^2}\right] \end{bmatrix} \begin{bmatrix} \psi_n^{(2)} \\ \psi_n^{(4)} \end{bmatrix} = (1+\nu)\lambda_n^2 \begin{bmatrix} \psi_n^{(2)} \\ \psi_n^{(4)} \end{bmatrix} \tag{3.14}$$

$$(1+\nu)D\psi_n^{(4)}(1) = D\psi_n^{(2)}(1) + \nu\psi_n^{(2)}(1) \tag{3.15}$$

$$\psi_n^{(4)}(1) = 0 \tag{3.16}$$

resulting in the biorthgonality

$$\langle \phi_m^{(2)}\psi_n^{(2)} + \phi_m^{(4)}\psi_n^{(4)}\rangle = 0 \text{ for } m \neq n. \tag{3.17}$$

In terms of the Bessel functions the two biorthogonal vectors are given by

$$\begin{bmatrix} \psi_n^{(1)}(r) \\ \psi_n^{(3)}(r) \end{bmatrix} = A_n \begin{bmatrix} -\lambda_n J_1(\lambda_n) r J_0(\lambda_n r) + [-2\nu J_1(\lambda_n) + \lambda_n J_0(\lambda_n)] J_1(\lambda_n r) \\ -\lambda_n J_1(\lambda_n) r J_0(\lambda_n r) + [2J_1(\lambda_n) + \lambda_n J_0(\lambda_n)] J_1(\lambda_n r) \end{bmatrix} \tag{3.18}$$

$$\begin{bmatrix} \psi_n^{(2)}(r) \\ \psi_n^{(4)}(r) \end{bmatrix} = B_n \begin{bmatrix} 2(1+\nu) J_1(\lambda_n) J_1(\lambda_n r) \\ \lambda_n J_1(\lambda_n) r J_0(\lambda_n r) - \lambda_n J_0(\lambda_n) J_1(\lambda_n r) \end{bmatrix} \tag{3.19}$$

where

$$A_n = \frac{1}{2(1+\nu)\lambda_n^2 J_1^2(\lambda_n) P(\lambda_n)} \tag{3.20}$$

$$B_n = \frac{1}{2(1+\nu)\lambda_n J_1^2(\lambda_n) P(\lambda_n)} \tag{3.21}$$

$$P(\lambda_n) = -\lambda_n^2 J_0^2(\lambda_n) + 2(1-\nu)\lambda_n J_0(\lambda_n) J_1(\lambda_n) - 2(1-\nu) J_1^2(\lambda_n) \tag{3.22}$$

and the normalising factors A_n and B_n have been introduced so that

$$\langle \phi_m^{(1)}\psi_n^{(1)} + \phi_m^{(3)}\psi_n^{(3)}\rangle = \delta_{mn} \tag{3.23}$$

$$\langle \phi_m^{(2)}\psi_n^{(2)} + \phi_m^{(4)}\psi_n^{(4)}\rangle = \delta_{mn}. \tag{3.24}$$

It is interesting to note that as in Spence[2] this formulation exhibits what might be called a "self-biorthogonality" where

$$\begin{bmatrix} \psi_n^{(1)} \\ \psi_n^{(3)} \end{bmatrix} = \frac{A_n}{(1+\nu)\lambda_n^4} \begin{bmatrix} -2\nu\phi_n^{(1)} + (1-\nu)\phi_n^{(3)} \\ (1-\nu)\phi_n^{(1)} + 2\phi_n^{(3)} \end{bmatrix} \tag{3.25}$$

and

$$\begin{bmatrix} \psi_n^{(2)} \\ \psi_n^{(4)} \end{bmatrix} = \frac{-B_n}{\lambda_n^3} \begin{bmatrix} \phi_n^{(4)} \\ \phi_n^{(2)} \end{bmatrix}. \tag{3.26}$$

In contrast, in the formulation of Little and Childs the (1, 3)-biorthogonal functions are given in terms of the (2, 4)-derived functions, and vice versa.

4. OPTIMAL WEIGHTING FUNCTIONS

In this section we consider the stress problem in which

$$\frac{\partial}{\partial r}(\sigma_{zz})_{z=0} \equiv f^{(1)}(r)$$

and (4.1)

$$(\sigma_{rz})_{z=0} \equiv f^{(2)}(r)$$

are prescribed functions of r. This does not fall into the class of canonical end problems categorised in Section 1. As in the case of the strip problem, we now seek weighting functions of the form

$$\chi_m^{(1)} = A\phi_m^{(1)} + B\phi_m^{(3)} \tag{4.2}$$

$$\chi_m^{(2)} = C\lambda_m^2\phi_m^{(2)} + D\lambda_m^2\phi_m^{(4)} \tag{4.3}$$

where A, B, C and D are constants to be determined. (The choice $A = -2\nu$, $B = (1-\nu)$, $C = 0$, $D = -(1+\nu)$ would produce the biorthogonal functions $\psi_m^{(1)}$, $\psi_m^{(2)}$ defined in Section 3, but as will be seen these are not optimal for the non-canonical problem.)

An infinite set of linear equations for the coefficients c_n in the derived expansions

$$f^{(1)} = \sum_n c_n\phi_n^{(1)} \tag{4.4}$$

$$f^{(2)} = \sum_n c_n\phi_n^{(2)} \tag{4.5}$$

is obtained by combining the quadratures of (4.4) times $\chi_m^{(1)}$ and (4.5) times $\chi_m^{(2)}$ for each m. This yields the set

$$\sum_n A_{mn}c_n = d_m \tag{4.6}$$

where

$$A_{mn} = \langle \chi_m^{(1)}\phi_n^{(1)} + \chi_m^{(2)}\phi_n^{(2)} \rangle \tag{4.7}$$

and

$$d_m = \langle \chi_m^{(1)}f^{(1)} + \chi_m^{(2)}f^{(2)} \rangle. \tag{4.8}$$

We now choose the constants A, B, C and D so as to make the off-diagonal elements of the matrix **A** as small as possible in absolute value compared with the diagonal elements. For this purpose the scalar products

$$\langle \phi_n^{(1)} \phi_m^{(1)} \rangle,\ \langle \phi_n^{(1)} \phi_m^{(3)} \rangle,\ \langle \phi_n^{(2)} \phi_m^{(2)} \rangle \text{ and } \langle \phi_n^{(2)} \phi_m^{(4)} \rangle \tag{4.9}$$

have been calculated explicitly. The expressions are cumbersome and are not given here,† but the salient feature is that the first three contain the factor $(\lambda_m^2 - \lambda_n^2)^{-3}$. As was noted by Spence[2] for the strip problem, the presence of any negative power of $(\lambda_m - \lambda_n)$ in the matrix A_{mn} leads to divergent row sum norms. The four constants A, B, C and D provide just sufficient freedom to suppress all such factors in the denominator.

The procedure for determining the optimal choice for the constants A, B, C and D involves taking the matrix elements (4.7) with $\chi_m^{(1)}$ and $\chi_m^{(2)}$ given by (4.2, 3), and dividing out the unwanted factors $(\lambda_m - \lambda_n)^{-1}$ giving three equations for the four constants.

Using the quadratures (4.9) the general matrix element A_{mn} may be written in the form

$$\begin{aligned}
&\frac{4\lambda_m^4\lambda_n^4 J_1(\lambda_m)J_1(\lambda_n)}{(\lambda_m+\lambda_n)^3(\lambda_m-\lambda_n)^3}\{[-(C-D(1+\nu))\lambda_m^5 + 2B(1+\nu)\lambda_m^4\lambda_n - (C+D(1+\nu))\lambda_m^3\lambda_n^2 \\
&\quad - (3A+3B\nu)\lambda_m^2\lambda_n^3 + (A+B\nu)\lambda_n^5]J_1(\lambda_m)J_0(\lambda_n) + \lambda_m^2[-(A+B\nu)\lambda_m^3 + (C-D(1+\nu))\lambda_m^2\lambda_n \\
&\quad + (3A-2B+B\nu)\lambda_m\lambda_n^2 + (C+D(1+\nu))\lambda_n^3]J_0(\lambda_m)J_1(\lambda_n)\} \\
&\quad + \text{a term subdominant for large } \lambda_m,\ \lambda_n.
\end{aligned} \tag{4.10}$$

The condition that the two terms multiplying the Bessel function products have a factor $(\lambda_m - \lambda_n)$ is the same, namely

$$A - B + C = 0, \tag{4.11}$$

and it turns out that we can divide out the other two factors $(\lambda_m - \lambda_n)$ from the above expression if the two equations

$$A + B\nu + D(1+\nu) = 0 \tag{4.12}$$

and

$$A + B\nu - C - D(1+\nu) = 0 \tag{4.13}$$

are satisfied, leaving a dominant term free of the undesirable factors $(\lambda_m - \lambda_n)$:

$$\frac{4(A+B\nu)\lambda_m^4\lambda_n^4 J_1(\lambda_m)J_1(\lambda_n)}{(\lambda_m+\lambda_n)^3}\{(3\lambda_m^2 + 3\lambda_m\lambda_n + \lambda_n^2)J_1(\lambda_m)J_0(\lambda_n) + \lambda_m^2 J_0(\lambda_m)J_1(\lambda_n)\}.$$

The three eqns (4.11, 12, 13) lead to the values

$$A = -(1-2\nu);\ B = -3;\ C = -2(1+\nu);\ D = 1. \tag{4.14}$$

The resulting weighting functions are thus

$$\chi_m^{(1)} = 2(1+\nu)\lambda_m^4[\lambda_m J_1(\lambda_m) r J_0(\lambda_m r) - \{J_1(\lambda_m) + \lambda_m J_0(\lambda_m)\} J_1(\lambda_m r)] \tag{4.15}$$

$$\chi_m^{(2)} = 2(1+\nu)\lambda_m^5[\lambda_m J_1(\lambda_m) r J_0(\lambda_m r) - \{J_1(\lambda_m) + \lambda_m J_0(\lambda_m)\} J_1(\lambda_m r)] \tag{4.16}$$

†More details of the calculation given below may be found in [6], obtainable from the author.

and the matrix elements are

$$A_{mn} = \frac{4(1+\nu)\lambda_m{}^4\lambda_n{}^4 J_1(\lambda_m)J_1(\lambda_n)}{(\lambda_m+\lambda_n)^3}\{(3\lambda_m{}^2+3\lambda_m\lambda_n+\lambda_n{}^2)J_1(\lambda_m)J_0(\lambda_n)+\lambda_m{}^2 J_0(\lambda_m)J_1(\lambda_n)\}$$

$$-\frac{4(1-\nu^2)\lambda_m{}^4\lambda_n{}^3 J_1{}^2(\lambda_m)J_1{}^2(\lambda_n)(3\lambda_m+\lambda_n)}{(\lambda_m+\lambda_n)^2} \tag{4.17}$$

$$A_{mm} = 2(1+\nu)\lambda_m{}^7 J_0(\lambda_m)J_1{}^2(\lambda_m)\{2\nu J_1(\lambda_m)+\lambda_m J_0(\lambda_m)\}. \tag{4.18}$$

It is interesting to note that although the constants A, B, C and D have been derived only from consideration of the dominant term (4.10), the subdominant term in the resulting matrix element (4.17) is also free of these factors. In order to see why this choice of constants should give rise to a more stable matrix, it is possible to use the eigenfunction quadratures (4.9) given explicitly in [6] and carry out an analysis of the asymptotic behaviour of the row sum norms $\Sigma_n |A_{mn}|/|A_{mm}|$ analogous to that given by Spence[2] for the strip problem. This shows that the matrix elements for Optimal Weighting Functions given above give rise to smaller row sum norms than for Unmodified Biorthogonal Weighting Functions, as shown by the results in Table 1.

5. DETAILS OF THE NUMERICAL RESULTS

In order to test the Optimal Weighting Functions (O.W.F.) derived in Section 4 and compare them with Unmodified Biorthogonal Weighting Functions (U.B.W.F.), a number of sample stress distributions were considered, and here we present results for two of these:

Case 1 $\sigma_{zz} = 1-2r^2$
$\sigma_{rz} = 0.$

Case 2 $\sigma_{zz} = 0$
$\sigma_{rz} = r.$

The first distribution is smooth, continuous and self-equilibrated, whereas the second distribution is incompatible with the edge conditions on $r = 1$. This second distribution presents a much more severe test than the first case, and in fact the decay of the coefficients is not sufficiently rapid for the partial sums to converge, and Cesaro sums have been summed instead (see Joseph[9], Joseph and Sturges[10]).

Three features of the numerical results obtained are worthy of note, showing the advantages offered by Optimal Weighting functions. These are

(1) The improvement in diagonal dominance of the truncated matrices.

(ii) The increase in stability of the earlier coefficients as the order of truncation is increased.

(iii) Improved convergence to the prescribed data for various orders of truncation.

The improvement in diagonal dominance of the truncated matrices can be seen in Table 1. Not only are the row sum norms less for Optimal Weighting Functions than for Unmodified Biorthogonal Weighting functions, but they are decreasing with the row index, and they are less

Table 1. $\Sigma_n |A_{mn}|/|A_{mm}|$ for $N = 50$. 100 and for various values of the row index m

N=50			N=100		
m	UBWF	OWF	m	UBWF	OWF
10	6.5801	3.7618	10	6.9351	4.6334
20	9.0141	2.9986	20	9.1039	3.8639
30	11.2604	2.5358	30	11.3648	3.3871
40	13.2192	2.2116	40	13.4944	3.0445
50	8.7778	1.9673	50	15.5056	2.7791
			60	17.4223	2.5640
			70	19.2549	2.3844
			80	20.9906	2.2313
			90	22.4725	2.0986
			100	14.2170	1.9820

Table 2(a). The first two coefficients for $\sigma_{zz} = 1 - 2r^2$ $\sigma_{rz} = 0$ using various values of N

Unmodified Biorthogonal Weighting Functions

N	$Re(c_1)$	$Im(c_1)$	$Re(c_2)$	$Im(c_2)$
5	-0.11902E-1	0.11986E-1	0.34817E-3	-0.16901E-3
10	-0.16985E-1	0.14775E-1	0.47215E-4	-0.25240E-3
20	-0.16622E-1	0.14578E-1	0.66895E-4	-0.24615E-3
50	-0.16582E-1	0.14556E-1	0.69037E-4	-0.24546E-3
100	-0.16566E-1	0.14547E-1	0.69904E-4	-0.24518E-3

Optimal Weighting Functions

N	$Re(c_1)$	$Im(c_1)$	$Re(c_2)$	$Im(c_2)$
5	-0.16470E-1	0.14509E-1	0.75161E-3	-0.24251E-3
10	-0.16588E-1	0.14566E-1	0.68057E-4	-0.24554E-3
20	-0.16572E-1	0.14549E-1	0.69270E-4	-0.24531E-3
50	-0.16558E-1	0.14543E-1	0.70286E-4	-0.24505E-3
100	-0.16557E-1	0.14542E-1	0.70409E-4	-0.24502E-3

Table 2(b). The first two coefficients for $\sigma_{zz} = 0$ $\sigma_{rz} = r$ using various values of N

Unmodified Biorthogonal Weighting Functions

N	$Re(c_1)$	$Im(c_1)$	$Re(c_2)$	$Im(c_2)$
5	-0.10644E+0	0.67713E-1	-0.68485E-2	-0.14244E-2
10	-0.31706E-1	0.26662E-1	-0.23732E-2	-0.20218E-3
20	-0.11428E-1	0.15645E-1	-0.12832E-2	0.14838E-3
50	0.83730E-2	0.48957E-2	-0.22527E-3	0.49061E-3
100	0.54865E-1	-0.20350E-1	0.22660E-2	0.12874E-2

Optimal Weighting Functions

N	$Re(c_1)$	$Im(c_1)$	$Re(c_2)$	$Im(c_2)$
5	0.27844E-1	-0.64030E-2	0.63268E-3	0.82867E-3
10	0.29602E-1	-0.72516E-2	0.77081E-3	0.85079E-3
20	0.31084E-1	-0.79308E-2	0.88376E-3	0.87495E-3
50	0.32631E-1	-0.86280E-2	0.99925E-3	0.90250E-3
100	0.33516E-1	-0.90251E-2	0.10648E-2	0.91860E-3

Table 3. Summed expansions for $\sigma_{zz} = 1 - 2r^2$ $\sigma_{rz} = 0$

Unmodified Biorthogonal Weighting Functions

r	$\sigma_{zz}=1-2r^2$	N=5	N=10	N=50	N=100
0.0	1.00	0.8283	1.0173	1.0006	1.0027
0.1	0.98	0.8834	0.9825	0.9803	0.9804
0.2	0.92	0.8936	0.9298	0.9204	0.9204
0.3	0.82	0.7422	0.8237	0.8204	0.8204
0.4	0.68	0.5895	0.6874	0.6803	0.6803
0.5	0.50	0.4723	0.5047	0.5003	0.5003
0.6	0.28	0.2426	0.2839	0.2802	0.2803
0.7	0.02	-0.0451	0.0241	0.0202	0.0203
0.8	-0.28	-0.2866	-0.2805	-0.2800	-0.2798
0.9	-0.62	-0.5559	-0.6254	-0.6203	-0.6200
1.0	-1.00	-0.7252	-1.0257	-1.0015	-0.0006

Optimal Weighting Functions

r	$\sigma_{zz}=1-2r^2$	N=5	N=10	N=50	N=100
0.0	1.00	1.0243	0.9794	1.0000	0.9997
0.1	0.98	0.9830	0.9880	0.9800	0.9800
0.2	0.92	0.9070	0.9135	0.9201	0.9200
0.3	0.82	0.8277	0.8264	0.8200	0.8199
0.4	0.68	0.6882	0.6744	0.6800	0.6799
0.5	0.50	0.4841	0.5056	0.5000	0.4999
0.6	0.28	0.2781	0.2754	0.2799	0.2799
0.7	0.02	0.0386	0.0239	0.0202	0.0199
0.8	-0.28	-0.2892	-0.2817	-0.2803	-0.2801
0.9	-0.62	-0.6262	-0.6213	-0.6195	-0.6201
1.0	-1.00	-1.0034	-0.9990	-0.9999	-1.0000

Table 4. Summed expansions for $\sigma_{zz} = 0$ $\sigma_{rz} = r$. Note that the partial sums for this distribution are not convergent, and the results given are Cesaro sums

Unmodified Biorthogonal Weighting Functions

r	$\sigma_{rz} = r$	N=5	N=10	N=50	N=100
0.0	0.0	0.0000	0.0000	0.0000	0.0000
0.1	0.1	0.6512	0.0311	0.3377	0.4175
0.2	0.2	0.2314	-0.0559	-0.1055	0.4858
0.3	0.3	-0.7526	-0.0084	0.3475	0.5616
0.4	0.4	-1.0633	-0.0862	0.0844	0.6277
0.5	0.5	-0.6903	-0.0698	0.3589	0.6841
0.6	0.6	-0.7798	-0.1509	0.2812	0.7432
0.7	0.7	-1.5204	-0.1756	0.3425	0.8352
0.8	0.8	-1.5147	-0.2722	0.4142	1.0034
0.9	0.9	-0.2523	-0.0649	0.4407	1.1914
1.0	1.0	0.0000	0.0000	0.0000	0.0000

Optimal Weighting Functions

r	$\sigma_{rz} = r$	N=5	N=10	N=50	N=100
0.0	0.0	0.0000	0.0000	0.0000	0.0000
0.1	0.1	0.0356	0.1071	0.0965	0.0967
0.2	0.2	0.1574	0.1754	0.1888	0.1926
0.3	0.3	0.3124	0.2875	0.2857	0.2884
0.4	0.4	0.3889	0.3565	0.3774	0.3839
0.5	0.5	0.4027	0.4673	0.4738	0.4790
0.6	0.6	0.4898	0.5248	0.5625	0.5734
0.7	0.7	0.6656	0.6414	0.6595	0.6667
0.8	0.8	0.7029	0.6588	0.7390	0.7579
0.9	0.9	0.4080	0.7860	0.8420	0.8412
1.0	1.0	0.0000	0.0000	0.0000	0.0000

subject to the effects of truncation. Although the truncated matrices are not strictly diagonally dominated as in the strip case [2], so that truncation is still not theoretically justified, the use of Optimal weighting functions can be seen from the results in Tables 2–4 to produce a much more satisfactory solution.

Tables 2(a–b) list the first two coefficients for varying orders of truncation (by the order of truncation we mean the number of *pairs* of eigenvalues used. Thus $N = 5$ means that a 10×10 system has been solved). The increased stability in these early coefficients is apparent—for the first distribution $\mathrm{Re}(c_1)$ calculated using OWF and listed in Table 2(a) only changes in the fourth decimal place as the order of truncation is increased from $N = 10$ to $N = 100$, whereas comparable accuracy is only obtained from UBWF in Table 2(b) when $N = 50$. For the incompatible distribution the behaviour of the coefficients obtained from UBWF is very erratic.

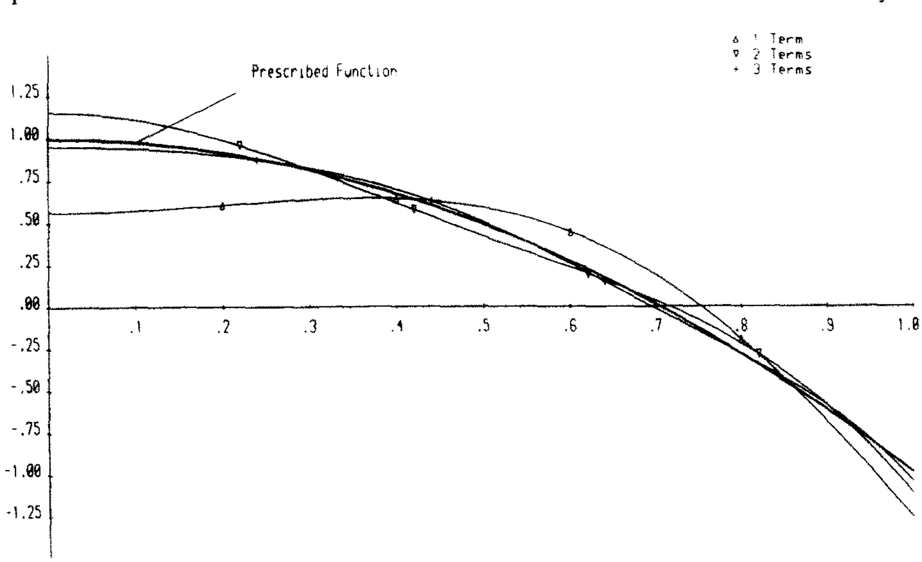

Fig. 2. Convergence of the expansion for $\sigma_{zz} = 1 - 2r^2$ when $N = 100$.

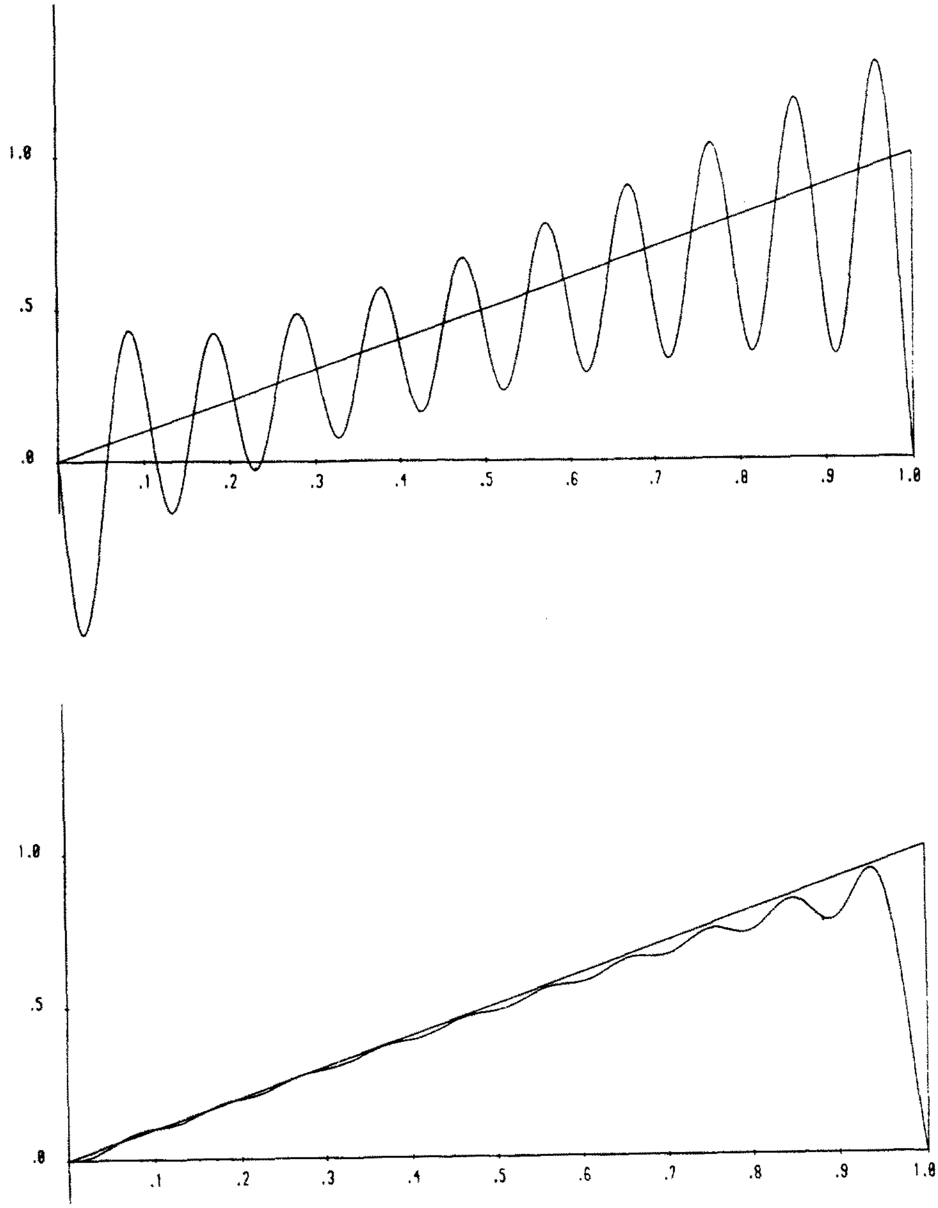

Fig. 3(a). Divergence of the partial sums for the incompatible stress distribution $\sigma_{rz} = r$ $\sigma_{zz} = 0$ with $N = 100$. Expansion summed to 20 terms. (b) Convergence of the Cesaro sums for $\sigma_{rz} = r$ showing a Gibbs' phenomenon near $r = 1$. $N = 100$. Expansion summed to 20 terms.

and even for $N = 100$ they do not seem to be approaching a limit, whereas for OWF the coefficients do appear to be converging, albeit more slowly than for case 1.

Tables 3 and 4 show the expansions summed for the two distributions evaluated on $z = 0$ and compared with the prescribed stress. The results shown in Table 3(a) for the smooth distribution $\sigma_{zz} = 1 - 2r^2$ are slightly more accurate for OWF than for UBWF, but as can be seen from the graph of the partial sums for this distribution in Fig. 2 the convergence is very rapid in any case. There is, however, a striking difference for the incompatible distribution σ_{rz}. It should be stressed again that the decay of the coefficients for this case is not sufficiently rapid for the series of partial sums to converge, and Fejer's method of summing Cesaro sums has been employed. This situation and its analogy with classical Fourier series is a very interesting point of analysis, and has been discussed recently by Joseph, Sturges and Warner[11]. The results obtained from UBWF do not seem to be converging, even when $N = 100$. On the other hand OWF are giving quite reasonable results when Cesaro sums are calculated, although as may be seen from Fig. 3(a–b) the Gibbs' phenomenon near $r = 1$ has not been entirely suppressed.

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